4/10/26 Collect HW

Recall:

If V is a subspace of $\mathbb{R}^n$ with orthonormal basis $\vec{u}_1, \ldots, \vec{u}_m$ then

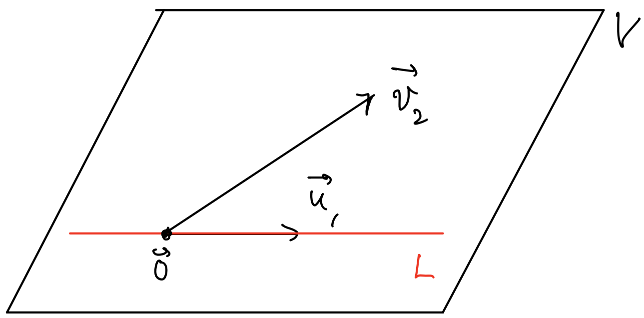
$$\mathrm{proj}_V(\vec{x}) = \vec{x}^{\parallel} \quad \text{(orthogonal projection)}$$
$$= (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_m \cdot \vec{x})\vec{u}_m$$

for all $\vec{x} \in \mathbb{R}^n$. We'll use this as part of the Gram-Schmidt process.

Now we're doing the Gram-Schmidt process (in a plane V so far).

We started with any basis $\{\vec{v}_1, \vec{v}_2\}$. We scaled $\vec{v}_1$ to unit length, $\vec{u}_1$

We set L = line spanned by $\vec{u}_1$.



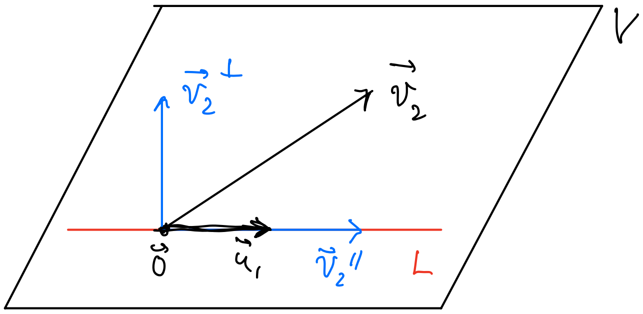
Then we decomposed $\vec{v}_2$ as

$$\vec{v}_2 = \vec{v}_2^{\parallel} + \vec{v}_2^{\perp}$$

where $\underbrace{\vec{v}_2^{\parallel} \text{ is parallel to } L}$ and $\underbrace{\vec{v}_2^{\perp} \text{ is perpendicular to } L}_{\text{this is the vector we're looking for now.}}$.

namely $\vec{v}_2^{\parallel} = \mathrm{proj}_L(\vec{v}_2)$

$$= (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1$$



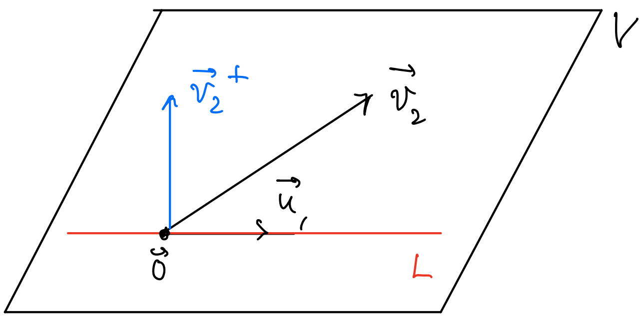
$\vec{v}_1, \vec{v}_2$ are orthogonal

$\vec{v}_2^{\parallel}$ is not necessarily equal to $\vec{u}_1$.
We don't need that.

So $\vec{v}_2^{\perp} = \vec{v}_2 - \vec{v}_2^{\parallel}$

$$= \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1$$

here's the current picture (we won't need $\vec{v}_2^{\parallel}$)



Now scale $\vec{v}_2^{\perp}$ to get $\vec{u}_2$ (perp. to $\vec{u}_1$ since it's perp. to L)

$$\vec{u}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|}\vec{v}_2^{\perp}$$

Then $\{\vec{u}_1, \vec{u}_2\}$ are our orthonormal basis for V.

For a higher-dim'l vector space you keep repeating this process:

<u>Gram-Schmidt Process</u>

Start with any basis $\{\vec{v}_1, \dots, \vec{v}_m\}$ of V in $\mathbb{R}^n$.

- $\vec{u}_1 = \dfrac{1}{\|\vec{v}_1\|} \vec{v}_1$

- $\vec{v}_2 = \vec{v}_2^{\,\parallel} + \vec{v}_2^{\,\perp} \longrightarrow \vec{v}_2^{\,\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\,\vec{u}_1$

  $\vec{u}_2 = \dfrac{1}{\|\vec{v}_2^{\,\perp}\|} \vec{v}_2$ Note this doesn't involve $\vec{v}_3$ yet.

Now let $L = \text{span}(\vec{u}_1, \vec{u}_2)$

- $\vec{v}_3 = \text{proj}_L(\vec{v}_3) + \vec{v}_3^{\,\perp} \longrightarrow \vec{v}_3^{\,\perp} = \vec{v}_3 - \left[(\vec{u}_1 \cdot v_3)\vec{u}_1 + (\vec{u}_2 \cdot v_3)\vec{u}_2\right]$

  $\vec{u}_3 = \dfrac{1}{\|v_3^{\perp}\|} \vec{v}_3^{\,\perp}$

etc.

<u>Ex</u> Let $\mathcal{B} = \left\{ \underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}} \right\}$ basis for $\mathbb{R}^3$.

Find an orthonormal basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ for $\mathbb{R}^3$ where $\vec{u}_1$ is a scalar multiple of $\vec{v}_1$.

$$\boxed{\vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{3}\vec{v}_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}}$$

$$\vec{v}_2^{\,\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\,\vec{u}_1$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \left(\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right) \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \left(\frac{1}{3} + \frac{4}{3} + \frac{6}{3} \right) \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \left(\frac{11}{3} \right) \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 11/9 \\ 22/9 \\ 22/9 \end{bmatrix} = \begin{bmatrix} -2/9 \\ -4/9 \\ 5/9 \end{bmatrix}$$

$$\|\vec{v}_2^{\perp}\| = \sqrt{4/81 + 16/81 + 25/81} = \sqrt{\frac{45}{81}} = \frac{3\sqrt{5}}{9} = \frac{\sqrt{5}}{3}$$

$$\boxed{\vec{u}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|} \vec{v}_2^{\perp} = \begin{bmatrix} \frac{-2}{9} \cdot \frac{3}{\sqrt{5}} \\ \frac{-4}{9} \cdot \frac{3}{\sqrt{5}} \\ \frac{5}{9} \cdot \frac{3}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} -2/3\sqrt{5} \\ -4/3\sqrt{5} \\ 5/3\sqrt{5} \end{bmatrix}}$$

$$\vec{v}_3^{\perp} = \vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2$$

$=$ (leave to you)

$$\vec{u}_3^{\perp} = \frac{1}{\|\vec{v}_3^{\perp}\|} \vec{v}_3^{\perp}$$

<u>Ex</u> Find the change of basis matrix to go from $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ to $\mathcal{C} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$.

We did this when one of the bases was the standard basis, but the general idea is the same. See Def 4.3.3 (which we didn't cover).

Note

$$\vec{v}_1 = 3\vec{u}_1$$

$$\vec{v}_2 = (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 + (\vec{v}_2 \cdot \vec{u}_2)\vec{u}_2$$

$$\left(\begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}1/3\\2/3\\2/3\end{bmatrix}\right)\begin{bmatrix}1/3\\2/3\\2/3\end{bmatrix} + \left(\begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}-2/3\sqrt{5}\\-4/3\sqrt{5}\\5/3\sqrt{5}\end{bmatrix}\right)\begin{bmatrix}-2/3\sqrt{5}\\-4/3\sqrt{5}\\5/3\sqrt{5}\end{bmatrix}$$

$$= \frac{11}{3}\vec{u}_1 + \frac{5}{3\sqrt{5}}\vec{u}_2$$

$$\vec{v}_3 = (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 + (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2 + (\vec{v}_3 \cdot \vec{u}_3)\vec{u}_3$$

$$= c\,\vec{u}_1 + d\,\vec{u}_2 + e\,\vec{u}_3 \quad \text{(say)}$$

Ch. of basis matrix:

$$\begin{bmatrix}3 & 11/3 & c\\0 & 5/3\sqrt{5} & d\\0 & 0 & e\end{bmatrix} = \left[\,[\vec{v}_1]_e \,\middle|\, [\vec{v}_2]_e \,\middle|\, [\vec{v}_3]_e\,\right]$$